Recall that GR has 2 parts:

1. Einstein's Equation tells us how sources create curved geometries.
2. Minimal-coupling, e.g. the geodesic equation for gravity only, tells us how curvature influences motion.

Let us consider a nontrivial (curved) solution of EE, the Schwarzschild metric:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

In our next lecture we will derive this using only the assumption of spherical symmetry. This geometry describes the exterior of any spherical source, e.g. planet, star, etc.

Now that we have a nontrivial curved geometry in hand, Schwarzschild we can explore (2) and in doing so we will find predictions of GR that differ from the Newtonian theory, i.e. ways of testing GR.

Since the Schwarzschild geometry describes gravity around a spherical source, let's review the relevant Newtonian analysis of the same scenario.

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Central forces in Newtonian Mechanics

$\vec{F} = f(r)\hat{r}$ w/ $\{r, \theta, \phi\}$

$\vec{\tau} = \vec{r} \times \vec{F} = 0 = \frac{d\vec{L}}{dt} \Rightarrow \vec{L} = \text{constant} \Rightarrow \begin{cases} \text{direction} \Rightarrow \text{motion is in 2D plane} \\ \text{magnitude} \Rightarrow m r^2 \dot{\phi} = \text{constant} \end{cases}$

we usually set $\theta = \frac{\pi}{2}$ and work w/ $\{r, \phi\}$

$\dot{\phi} = \frac{L}{m r^2}$

System = source (M) + test object (m_t)

$E_{tot} = \frac{1}{2} m_t \vec{v} \cdot \vec{v} + V(r) = \text{constant}$

$= \frac{1}{2} m_t \dot{r}^2 + \frac{1}{2} m_t r^2 \dot{\phi}^2 + V(r)$

$= \frac{1}{2} m_t \dot{r}^2 + \underbrace{\frac{L^2}{2 m_t r^2} + V(r)}$

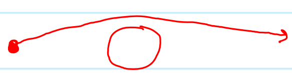
$E_{tot} = \frac{1}{2} m_t \dot{r}^2 + V_{eff}(r)$ DiffEQ for 1D motion in $\{r\}$

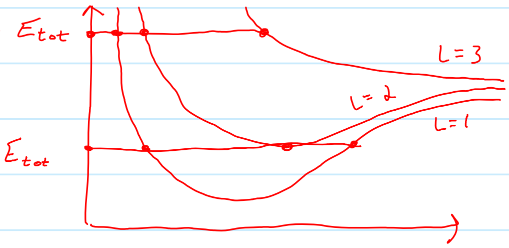
henceforth we will set $m_t = 1$ (when $m_t > 0$)

We can get the behavior by plotting $V_{eff}(r)$. Note: Different choices of L give different $V_{eff}(r)$.

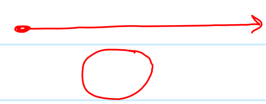
$V_{eff}(r) = \frac{L^2}{2r^2} - \frac{GM}{r}$

All cases scatter, i.e.

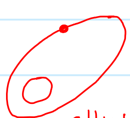




Note: If $m_t = 0$, $V(r) = 0$

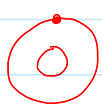


L=1



elliptical orbit

L=2



circular orbit

For $m_t > 0$, if the total energy is high enough then we have scattering for all L. If the total energy is low enough we get bound states (orbits) for small enough L. These are "stable" in the sense that if we perturb them, they only change slightly. For $m_t = 0$ we always "scatter"!

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Geodesic Motion in the Schwarzschild Geometry

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 \Rightarrow g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2GM}{r}\right) & & & \\ & \left(1-\frac{2GM}{r}\right)^{-1} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix}$$

To proceed in a manner analogous to the Newtonian approach note that $g_{\mu\nu}$ is independent of t, ϕ.
Then $K_t^\mu = (1,0,0,0)$ and $K_\phi^\mu = (0,0,0,1)$ are Killing vectors.

$$K_{t\mu} = \left(-\left(1-\frac{2GM}{r}\right), 0, 0, 0\right) \qquad K_{\phi\mu} = (0,0,0, r^2\sin^2\theta) = (0,0,0,r^2)$$

$\uparrow$ setting $\theta = \frac{\pi}{2}$

The conserved "momenta" are:

$$K_{t\mu}\frac{dx^\mu}{d\lambda} = -\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right) = E$$

$$\underbrace{K_{\phi\mu}\frac{dx^\mu}{d\lambda}}_{} = r^2\frac{d\phi}{d\lambda} = L$$

recall $m_t = 1$

Note that we have no $\frac{dr}{d\lambda}$ term. However we can show that for geodesics:

$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \Rightarrow \frac{D\epsilon}{d\lambda} = -g_{\mu\nu}\left[\underbrace{\left(\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda}\right)}_{=0}\frac{dx^\nu}{d\lambda} + \frac{dx^\mu}{d\lambda}\underbrace{\left(\frac{D}{d\lambda}\frac{dx}{d\lambda}\right)}_{=0}\right]$$

(used metric compatibility)

$=0 \leftarrow$ geodesic equation $\rightarrow =0$

Therefore ϵ is a conserved quantity for geodesics.

$$\text{But: } \epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

$$= E^2\Big/\left(1-\frac{2GM}{r}\right) - \left(\frac{dr}{d\lambda}\right)^2\Big/\left(1-\frac{2GM}{r}\right) - L^2/r^2$$

Rearranging:

$$\frac{1}{2}E^2 = \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \underbrace{\frac{1}{2}\epsilon - \frac{GM}{r}\epsilon + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}}_{V_{eff}(r)}$$

Recall that for $m_t > 0$ w/ $\lambda = \tau$ $\quad \epsilon = -U_\mu U^\mu = +1$

$m_t = 0$ $(ds^2 = 0)$ $\quad \epsilon = 0$

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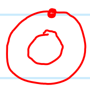
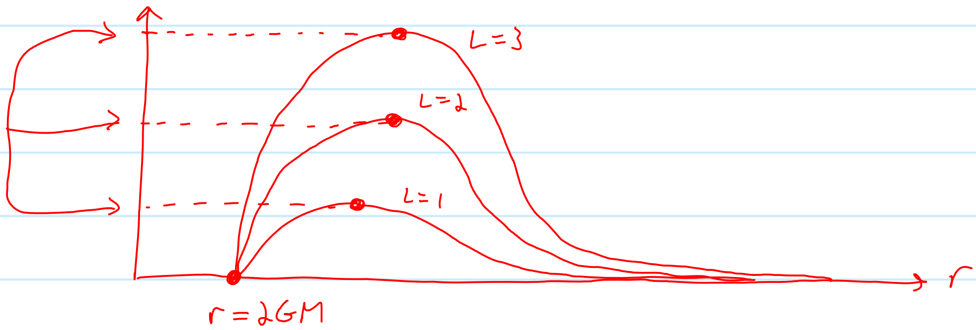
$M_t = 0$

$$V_{eff}(r) = \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$$

Unstable Circular Orbits



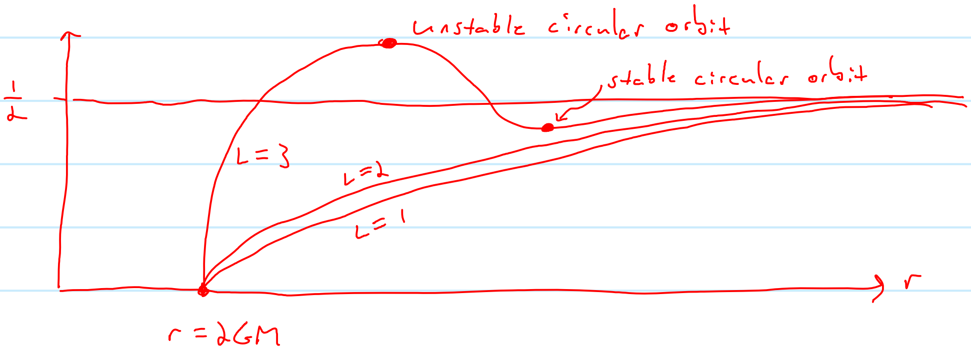


This is certainly a different prediction than the  Newtonian result!

$m_t > 0$

Note: This is the GR correction to the Newtonian result

$$V_{eff}(r) = -\frac{GM}{r} + \frac{L^2}{2r^2} \overbrace{- \frac{GML^2}{r^3}}$$



If L is large enough then we have a stable/unstable pair of circular orbits.
But if L is too small then no stable circular orbits exist, which can be turned into a minimum radius condition:

$$\left.\frac{dV_{eff}(r)}{dr}\right|_{r_c} = 0 \Rightarrow r_{c\pm} = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2L^2}}{2GM} \quad \text{w/ } r_{c+} = \text{stable}, \; r_{c-} = \text{unstable}$$

The smallest possible radius is when $r_{min} = r_{c+} = r_{c-} = \frac{L^2}{2GM} = 6GM$, but Newton predicts stable circular orbits for any r!
So to test GR we can just look for the instability of circular orbits for $r < r_{min}$.

But... Earth :	$6\left(\frac{1}{c^2}\right)GM_E \sim 0.03$ m	$R_E \sim 6 \times 10^6$ m	r_{min} is inside Earth!
Sun :	$6\left(\frac{1}{c^2}\right)GM_S \sim 8850$ m	$R_S \sim 7 \times 10^8$ m	r_{min} is inside Sun!
M_S White Dwarf :	~ 8850 m	$R_{WD} \sim 10^6$ m	crap!
M_S Neutron Star :	~ 8850 m	$R_{NS} \sim 10^4$ m	damn!
⋮			
M_S Black Hole :	~ 8850 m	$R_{BH} \sim 0$	Sweet!

In case you are worried, the event horizon is at $r = 2GM$, so r_{min} is well outside.

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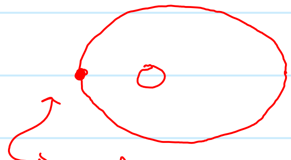
Perihelion Shift

A more practical means of testing GR is to use existing astronomical data.

Consider:



Point of closest approach (perihelion)

Newtonian Case: $E = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} + \frac{L^2}{2r^2}$

Using $L = r^2 \frac{d\phi}{dt}$ to replace $dt \to \frac{r^2}{L} d\phi$ we get

$\left(\frac{dr}{d\phi}\right)^2 - \frac{2GM}{L^2} r^3 + r^2 = \frac{2E}{L^2} r^4$ w/ solution $r(\phi) = \frac{L^2}{GM(1 + e\cos\phi)}$

$e(G, M, L)$

Due to $\cos(\phi)$ we have: $r(\phi + 2\pi) = r(\phi)$, i.e. perihelion does not change!

General Relativity: Using the GR corrected $V_{eff}(r)$ and a similar rewriting one can show that to leading order $r(\phi) = \frac{L^2}{GM(1 + e\cos[(1-\alpha)\phi])}$

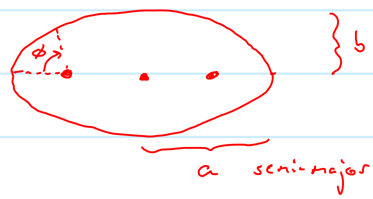
$\alpha = \frac{3G^2M^2}{L^2}$

$\cos[(1-\alpha)\phi] = \cos\left[(1-\alpha)\left(\phi + \frac{2\pi}{1-\alpha}\right)\right] \Rightarrow$ periodicity in $\frac{2\pi}{1-\alpha} \simeq 2\pi(1+\alpha) = 2\pi + 2\pi\alpha$

Then $\Delta\phi = 2\pi\alpha = \frac{6G^2M^2\pi}{L^2}$ is the angular shift after one orbit.

To get a numerical value we need L which we can get by measuring the semi-major and minor axes of the orbit.

$$r(\phi) = \frac{(1-e^2)a}{1+e\cos\phi}$$ where



b semi-minor

$$e = \sqrt{1-\frac{b^2}{a^2}}$$

Compare to the Newtonian (leading order) case:

$$r(\phi) = \frac{L^2}{GM(1+e\cos\phi)} \Rightarrow \frac{L^2}{GM} = (1-e^2)a \Rightarrow \boxed{\Delta\phi = \frac{6\pi GM}{(1-e^2)a}}$$

For our solar system $M = M_s$, so largest $\Delta\phi$ is for smallest a, i.e. Mercury!

For Mercury: $a = 5.79 \times 10^{10}\,m$
$e = 0.2056$
$T = 88$ days

$$\Delta\phi = 43 \frac{\text{arcseconds}}{\text{century}}$$

But the observed value is $\Delta\phi = 5601 \frac{\text{arcseconds}}{\text{century}}$!

However taking into account precession of equinoxes, gravitational pull of other planets and oblateness of the sun we get $5558 \frac{\text{arcseconds}}{\text{century}}$ from these alone.

$5558 + 43 = 5601$ BAM! Go Einstein!!